Threshold effects on synchronization of pulse-coupled oscillators

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The threshold effects of synchronous firing are studied in a class of models. The main result is that for almost all initial conditions, the oscillation system approaches synchrony when the threshold condition is below a critical value.

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I. INTRODUCTION

Mutual synchronization is known to exist in nature. Examples includes crickets that chirp in unison and fireflies that flash in synchrony. These phenomena all start with arbitrary initial conditions and end up in a mutually synchronized state. In his seminal work, Winfree [1] found that mutual synchronization can occur when oscillators within a system are strongly attracted to their limit cycles. A recent proof has been given by Mirollo and Strogatz [2] on the problem of synchronization of the "integrate and fire" (IF) oscillators. They show quite generally that in a class of "all to all" coupled IF oscillator systems, mutual synchronization arises in almost all initial conditions. (Here, "almost all" means up to a set of Lebesgue measure zero.) Their work is based on a generalized version of Peskin's model [3] which assumes that the oscillators are identical and each is coupled to all other oscillators. However, these basic assumptions certainly make the model less realistic. Since in any real system, such as fireflies flashing in synchrony, synchronization is only a result of average. (Here, "average" means that it is only a large portion of the population which is flashing in synchrony.) As a matter fact, fireflies are not identical and furthermore, not every firefly responds to all the others. Therefore, it would be interesting to see whether or not synchronization can arise when the above assumptions are dropped.

In this work, we provide a partial answer to the case where the "all to all" condition is removed by introducing a threshold condition on the coupling between oscillators. In the model of [2], the oscillators are of the integrate and fire type and each described by a variable $x_i(t)(i=1,\ldots,N)$. When $x_i(t)=1$, the oscillator "fires" and $x_i(t^+)$ is set to zero. The other oscillators are then coupled through this firing process only in the following: when the ith oscillator "fires," it pulls all the others up by an amount ε, namely, $x_j(t)\rightarrow x_j(t)+\varepsilon$, for all $j\neq i$ and $x_j(t^+)=\min(x_j(t)+\varepsilon,1)$. In our work, we modify the above coupling condition by introducing a threshold x_c. We require that, when firing occurs, the other oscillators are pulled up only if their value x_j is greater than x_c. Therefore, not every oscillator is pulled during firing. Our main result is that by assuming that $x_i(t)$ is monotonic and concave downward, there exists a $\bar{x}_c$ such that when $x_c<\bar{x}_c$, the oscillators become synchronized for all N and for almost all initial conditions. Our proof is based on an extension of the method that was employed by Mirollo and Strogatz. In the next section, we describe our model in detail and discuss the synchronization of two oscillators. The problem of N oscillators is addressed in Sec. III. The final section contains the conclusions and a few comments are made.

II. TWO OSCILLATORS

Model

We will assume that all oscillators have identical dynamics and that the state variable $x_i(t)$ of each oscillator is a monotically increasing function. Furthermore, the maximum value of $x(t)$ is scaled to 1, i.e., $x(t)\in[0,1]$. To simplify our problem, we follow the approach of [2] and assume that x evolves as $x=f(\phi)$ with f being $[0,1]\rightarrow[0,1]$. f is a continuous, monotonic, increasing, and concave downward function of ϕ. Here, ϕ is a phase variable which is related to the period T of the oscillator, $d\phi/dt=1/T$. We also set $f(0)=0$ and $f(1)=1$. Our assumptions imply that $f'>0$ and $f''<0$. A particular example of this type of oscillator is Peskin's model with $f(\phi)=(1-e^{-\beta\phi})/(1-e^{-\beta})$. Since f is monotonic, the inverse of f exists. We denote $g=f^{-1}$ and $g(x)=\phi$. From the assumed properties of f, the function g is also monotonic increasing but concave upward: $g'>0$ and $g''>0$. The end points of g are also fixed: $g(0)=0$ and $g(1)=1$.

Due to the fact that f is monotonic, the threshold condition x_c can also be specified as $\phi_c=g(x_c)$ which is uniquely determined. From now on, we will use ϕ_c to specify the threshold effects. To discuss the synchronization of the two oscillators A and B, let us consider the return map $R(\phi)$. Suppose that the oscillator A has just fired, thus its phase ϕ_A is set to zero and the oscillator B has a phase $\phi_B=\phi$ (see Fig. 1). The return map $R(\phi)$ is then defined to be the phase of B immediately after the next firing of A. It is easy to see from Fig. 1 that, as ϕ_B starts from ϕ and moves toward 1, ϕ_A moves from zero to $1-\phi$. Therefore we define a firing map which determines the phase of A immediately after B fires,

$$h(\phi)=\begin{cases}1-\phi\ , & \phi>1-\phi_c\\ g(\varepsilon+f(1-\phi))\ , & \phi\leq 1-\phi_c\ .\end{cases} \tag{2.1}$$

The first condition in (2.1) corresponds to the fact that A

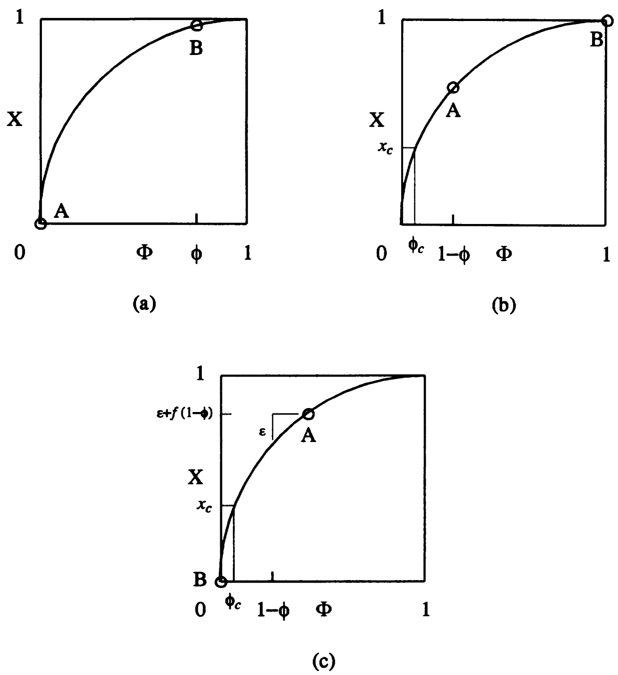

FIG. 1. Two oscillators governed by $x=f(\phi)$ and coupled by the rule (2.1). (a) The phases of the system right after oscillator A has fired. (b) The state of the system before B fires. (c) The state of the system just after B has fired. We have assumed $1-\phi>\phi_c$.

is not pulled up by the firing process of B (see Fig. 2).

The return map $R(\phi)$ is given in terms of h,

$$R(\phi)=h(h(\phi)) . \tag{2.2}$$

Let us observe that ϕ_c falls into two separate classes: (a) $\phi_c>\frac{1}{2}$ and (b) $\phi_c\leq\frac{1}{2}$. For the first case, it can be shown that there exists a window in which $R(\phi)=\phi$ and as a result there is no synchronization.

Theorem 1 . For $\phi_c>\frac{1}{2}$, if $\phi\in(1-\phi_c,\phi_c)$, then $R(\phi)=\phi$.

Proof. Since $\phi\in(1-\phi_c,\phi_c)$, we have $\phi<\phi_c$ which implies $1-\phi>1-\phi_c$. The firing map $h(\phi)$ and $R(\phi)$ is given by $h(\phi)=1-\phi$ and $R(\phi)=h(h(\phi))=h(1-\phi)=\phi$, respectively.

Hence, to be able to have synchronization for two oscillators for almost all initial conditions, we have to impose $\phi_c<\frac{1}{2}$. It is obvious that ϕ falls into three separate domains: (a) $\phi>1-\phi_c$, (b) $\phi<\phi_c$, and (c) $\phi_c\leq\phi\leq1-\phi_c$. It is straightforward to show that, for both cases (a) and (b), the system ends up in synchrony. This is due to the fact that $R(\phi)$ has no fixed point within these regions. First, we show that if $\phi_c<\frac{1}{2}$ and $\phi>1-\phi_c$, $R'(\phi)$ is always greater than 1. This can be seen as follows: if $\phi_c<\frac{1}{2}$ and $\phi>1-\phi_c$ then we have $1-\phi<1-\phi_c$. As a result, $R(\phi)=g(\varepsilon+f(\phi))$ and $R'(\phi)=g'f'$, hence $R'=g'(\varepsilon+f(\phi))/g'(f(\phi))$ where we have used the fact that $f'=1/g'$. Since $g''>0$, we have $R'>1$ as claimed. By using $R'>1$, we proceed to show the following theorem.

Theorem 2. If $\phi_c<\frac{1}{2}$ and $\phi>1-\phi_c$, then $R(\phi)$ has no fixed point and $R(\phi)>\phi$.

Proof. Since we have shown $R(\phi)=g(\varepsilon+f(\phi))$, together with $g'>0$, it is obvious that $R(\phi)>\phi$. This fact indicates that $R(\phi)$ approaches 1 if $R(\phi)$ has no fixed point. Let us consider the function $K(\phi)=\phi-R(\phi)$. The domain of $R(\phi)$ is $D=[1-\phi_c,g(1-\varepsilon)]$, where $\phi<g(1-\varepsilon)$ comes from the requirement that $R(\phi)<1$, otherwise we already have synchronization.

It is easy to see that the domain is not empty. If $g(1-\varepsilon)<1-\phi_c$ were true, it implies $\varepsilon>1-f(1-\phi_c)$ and hence $R(\phi)=g(\varepsilon+f(\phi))>g(1+f(\phi)-f(1-\phi_c))$, which is greater than 1 and leads to contradiction. Therefore we have $1-\phi_c<g(1-\varepsilon)$. This result also imposes a constraint on ε and ϕ_c, namely, $\varepsilon<1-f(1-\phi_c)$.

We can now proceed to show that $R(\phi)$ has no fixed point. Let us consider the values of K at the end points of D,

$$K(1-\phi_c)=1-\phi_c-g(\varepsilon+f(1-\phi_c)) ,$$

$$K[g(1-\varepsilon)]=g(1-\varepsilon)-1 .$$

It is obvious that $K(1-\phi_c)<0$ and $K[g(1-\varepsilon)]<0$. The derivative of K is

$$K'=1-R'=1-\frac{g'(\varepsilon+f(\phi))}{g'(f(\phi))} . \tag{2.3}$$

As we have shown earlier that $R'>1$, hence $K'<0$ and does not vanish inside D. Therefore, K does not vanish in D and R has no fixed point.

For the case where $\phi<\phi_c$, we can proceed in a similar fashion and show that $R(\phi)$ has no fixed point either. We state the result as a theorem.

Theorem 3. If $\phi_c<\frac{1}{2}$ and $\phi<\phi_c$, then $R(\phi)<\phi$ and $R(\phi)$ has no fixed point.

Proof. First we show that, for this case, $R'(\phi)>1$. It is easy to see that

$$h(\phi)=g(\varepsilon+f(1-\phi)) \tag{2.4}$$

and

$$R(\phi)=1-g(\varepsilon+f(1-\phi)) . \tag{2.5}$$

Hence,

$$R'(\phi)=g'f'=\frac{g'(\varepsilon+f(1-\phi))}{g'(f(1-\phi))}>1 ,$$

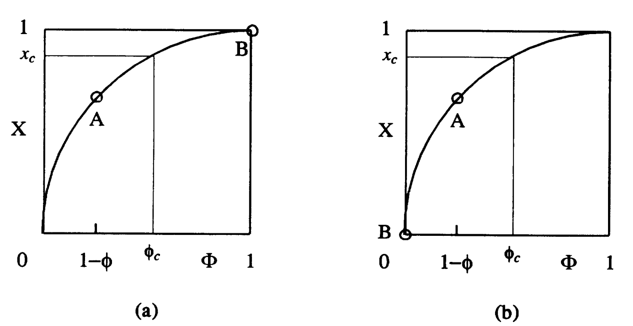

FIG. 2. Two oscillators with a large ϕ_c. (a) The state of the system right before B fires. (b) The state of the system just after B has fired. Since $\phi_c>1-\phi$, the phase of A is not pulled up by the firing process.

where we have used $g''>0$. From (2.5), it is obvious that $R(\phi)<\phi$. Let us consider the function $K(\phi)=\phi-R(\phi)$. As in the previous theorem, we will show that K does not vanish in its domain which, in this case, is $D=(\delta,\phi_c)$. Here δ is given by $1-g(1-\varepsilon)$ which comes from the fact that $h(\phi)$ does not synchronize. The values of K at the end points of D are

$$K(\delta)=1-g(1-\varepsilon)>0 ,$$
$$K(\phi_c)=g(\varepsilon+f(1-\phi_c))-(1-\phi_c)>0 .$$

The derivative of K is

$$K'=1+\frac{g'(\varepsilon+f(1-\phi))}{g'(f(1-\phi))} \tag{2.6}$$

and is obviously greater than 1. Hence we conclude that K is nonvanishing which implies that R has no fixed point as claimed.

The remaining case where $\phi\in(\phi_c,1-\phi_c)$ is different from the previous discussions because $R(\phi)$ has a fixed point in this case. Fortunately, the fixed point is a repeller and its proof is very similar to the one given by Mirollo and Strogatz [2].

To show that $R(\phi)$ has a repelling fixed point, we first show that the domain of $h(\phi)$ is given by $(\delta_1,1-\phi_c)$, where $\delta_1\equiv[1-g(f(1-\phi_c)-\varepsilon)]>\phi_c$. This can be seen by noting that if $h(\phi)=g(\varepsilon+f(1-\phi))>1-\phi_c$ then $R(\phi)=[1-g(\varepsilon+f(1-\phi))]<\phi_c$ and according to theorem 3 the system will synchronize. Hence we only need to consider $g(\varepsilon+f(1-\phi))<1-\phi_c$ which implies $\phi>[1-g(f(1-\phi_c)-\varepsilon)]=\delta_1$.

However, the domain D of $R(\phi)$ is only a subset of $(\delta_1,1-\phi_c)$ [it is easy to see that when $R(\phi)<\delta_1$ or $R(\phi)>1-\phi_c$, according to the previous theorem, the system will synchronize] and it can be shown that $D=(\delta_1,\delta_2)$ with δ_2 given by $h^{-1}(\delta_1)$. There is a possibility that $\delta_2<\delta_1$ and as a result D is an empty set. If $\delta_2<\delta_1$ then $h(h(\delta_2))<h(h(\delta_1))$ (this is from the fact that g is a monotonic increasing function). Explicitly we have

$$h(h(\delta_2))=h(\delta_1)=1-\phi_c ,$$
$$h(h(\delta_1))=g(\varepsilon+f(\phi_c)) .$$

Thus, $\delta_2<\delta_1$ implies

$$g(\varepsilon+f(\phi_c))>1-\phi_c .$$

This is equivalent to, say, $\varepsilon>f(1-\phi_c)-f(\phi_c)$ and hence $h(\phi)=g(\varepsilon+f(1-\phi))>1-\phi_c$. However, $h(\phi)>1-\phi_c$ is just the conclusion that we obtained in the previous paragraph where synchronization occurs. Similarly, one can show that if $\varepsilon>f(1-\phi_c)-f(\phi_c)$, then $\delta_1>\delta_2$. Thus we only have to consider $\delta_1<\delta_2$.

We can now proceed to show that there is a fixed point within D. The proof is the same as given in [2]. We introduce a function $F(\phi)\equiv\phi-h(\phi)$. It is not hard to see that a fixed point of h is also a fixed point of $R(\phi)$. Let us calculate $F(\delta_1)$ and $F(\delta_2)$ as

$$F(\delta_1)=\delta_1-h(\delta_1)=\phi_c-g(f(1-\phi)-\varepsilon) ,$$
$$F(\delta_2)=\delta_2-h(\delta_2)$$
$$=\delta_2-\delta_1>0$$

if $F(\delta_1)>0$, then $\phi_c>g(f(1-\phi_c)-\varepsilon)$ and it is equivalent to $\varepsilon>f(1-\phi_c)-f(\phi_c)$, but this is a contradiction to the fact that $\delta_2>\delta_1$. Therefore $F(\delta_1)$ must be negative. Thus, $F(\phi)=0$ has a root which is determinated by $g(\phi^*)=\phi^*$.

From $F'(\phi)=1-h'(\phi)$ and $h'(\phi)=-g'(\varepsilon+f(1-\phi))/g'(f(1-\phi))$, noting that $g''>0$ we have $F'(\phi)>2$. Hence $F(\phi)$ only crosses the ϕ axis once. Similarly, $R'(\phi)=h'(g)h'(\phi)>1$ which also implies that ϕ^* is also a unique fixed point of R and we have

$$R(\phi)>\phi \quad \text{if } \phi>\phi^* ,$$
$$R(\phi)<\phi \quad \text{if } \phi<\phi^* .$$

Therefore ϕ^* is a repeller. We have shown in this section that there exists a critical threshold value $\bar{x}_c=f(\frac{1}{2})$, such that when $x_c<\bar{x}_c$ a two-oscillator system synchronizes for almost all initial conditions.

III. N OSCILLATORS

Note that, since $\phi_c>\frac{1}{2}$ does not lead to synchronization for the case of two oscillators, it is not possible to have synchrony for $N\geq 2$. Therefore, we only concentrate on the case where $\phi_c<\frac{1}{2}$.

Since the oscillators are identical, the set of phases of N oscillators can be presented by the set

$$S=\{(\phi_1,\ldots,\phi_{N-1})\in R^{N-1} \text{ with } 0<\phi_1<\phi_2<\cdots<\phi_{N-1}<1\} , \tag{3.1}$$

where we have put $\phi_0=0$. Let $\phi=(\phi_1,\ldots,\phi_{N-1})$ be the set of phases right after a firing, and the firing map h is defined as the product of two operations. The first operation, which produces the phase right before the next firing, is given by the affine mapping σ: $R^{N-1}\to R^{N-1}$:

$$\sigma(\phi_1,\ldots,\phi_{N-1})=(1-\phi_{N-1},\phi_1+1-\phi_{N-1},\ldots,\phi_{N-2}+1-\phi_{N-1})=(\sigma_1,\ldots,\sigma_{N-1}) . \tag{3.2}$$

After the firing occurs, the new phases are obtained by the map τ where

$$\tau(\sigma_1,\ldots,\sigma_{N-1})=(g(\sigma_1),\ldots,g(\sigma_{N-1})) \tag{3.3}$$

where g is given by

$$g(\sigma)=\begin{cases}\sigma , & \phi_c>\sigma\\ g(\varepsilon+f(\sigma)) , & \phi_c\leq\sigma .\end{cases} \tag{3.4}$$

The firing map h is then given by

$$h(\phi)=\tau(\sigma(\phi)) \tag{3.5}$$

which describes the phases right after firing. It is obvious that S is invariant under σ, but not under τ. This is due to the fact that $f(\sigma_{N-1})+\varepsilon \geq 1$ is possible. When this happens, the oscillator $N-1$ has brought the oscillator $N-2$ together with it and they move together as one; the number of oscillators is then reduced from N to $N-1$. This event is called an absorption.

Because of the existence of absorption, the domain of h is not all of S. In fact, the domain is the set

$$S_\varepsilon = \{(\phi_1, \ldots, \phi_{N-1}) \in S \text{ such that } f(\phi_{N-2}+1-\phi_{N-1})+\varepsilon < 1\} . \tag{3.6}$$

Note that, due to the absorption process, the oscillators move together as a single oscillator, which has a pulse strength ε being the sum of the pulse strengths of the individual oscillator. Thus it is necessary to allow nonidentical pulse strengths in our discussions. It turns out that synchronization does not require identical pulse strengths for all oscillators.

First, we state a few facts about the firing map h and the return map $R = h_N h_{N-1} \ldots h_1$, where h_i means that the oscillator strength is ε_i. Let us consider a particular type of initial conditions where all $\phi_i (i=1\text{–}N-1)$ are less than ϕ_c and are denoted by

$$\phi_p = \{(\phi_1, \phi_2, \ldots, \phi_{N-1}) \text{ where } \phi_1 < \phi_2 < \cdots < \phi_{N-1} < \phi_c\} .$$

It is obvious that the firing map h implies

$$h_1(\phi_p) = \{g(\varepsilon_1 + f(1-\phi_{N-1})), g(\varepsilon_1 + f(1-\phi)N-1+\phi_1)), \ldots, g(\varepsilon_1 + f(1-\phi_{N-1}+\phi_{N-2}))\} .$$

It is easy to see that the return map

$$R(\phi_p) = \{(\phi_1', \phi_2', \ldots, \phi_{N-1}') \text{ such that } \phi_1' < \phi_2' < \cdots < \phi_{N-1}' < \phi_c\}$$

where $\phi_i' < \phi_i$ if at least one of the $\varepsilon_i \neq 0$. It is clear that R does not have a fixed point and these kind of initial conditions always leads to synchronization.

For the case where $\phi_p' = \{(\phi_1', \ldots, \phi_{N-1}') \text{ with } 1-\phi_c < \phi_1 < \phi_2, \ldots, < \phi_{N-1}\}$, we have

$$h^{N-1}(\phi_p') = \{(\phi_1', \phi_2', \ldots, \phi_{N-1}') \text{ with } \phi_1' < \phi_2' < \cdots < \phi_{N-1}' < \phi_c\} \tag{3.7}$$

and as a result of the previous paragraph, we conclude that ϕ_p' also ends up in synchrony. By using these facts of the return map, we have the following theorem.

Theorem 4. Let A_i be the set of initial conditions that will have at least i firings before an absorption occurs, namely,

$$A_i = \{\phi \in S \text{ such that } \phi \in S_{\epsilon_1}, h_1(\phi) \in S_{\epsilon_2}, h_2 h_1(\phi) \in S_{\epsilon_3}, \ldots, h_{i-1} h_{i-2} \ldots h_1(\phi) \in S_{\epsilon_i}\}$$

and $A = \bigcap_{n=1}^{\infty} A_i$, which is the set of initial conditions that live forever without any absorptions. Then A has the Lebesgue measure zero.

Proof. It is obvious that A is invariant under R, i.e., $R(A) \subset A$ and R is one to one on its domain. Let us consider the Jacobian determinant of R

$$\det(DR) = \prod_{i=1}^{N} \det(Dh_i) = \prod_{i=1}^{N} \det(D\tau_i)\det(D\sigma) . \tag{3.8}$$

By using the fact that $\sigma^N = I$ we have

$$|\det(DR)| = \left| \prod_{i=1}^{N} \det(D\tau_i) \right| . \tag{3.9}$$

The determinant of $D\tau_i$ is given by

$$\det(D\tau_i)|_\sigma = \prod_{k=1}^{N-1} g'f'(\sigma_k) . \tag{3.10}$$

For any $(\sigma_1, \sigma_2, \ldots, \sigma_{N-1})$, there is a possibility where $\det(D\tau_i)|_\sigma = 1$. This is corresponding to the case where $\sigma_k < \phi_c$ for all k and resulting in $g' = g'(f(\sigma_k))$, hence $g'f' = g'(g')^{-1} = 1$ as shown in Sec. II. However, this possibility is excluded from A since such σ implies synchronization which is contradicted to the assumption of A. Hence at least one factor on the right-hand side of (3.10) is not 1,

$$g'f'(\sigma_k) = \frac{g'(\varepsilon_i + f(\sigma_k))}{g'(f(\sigma_k))} . \tag{3.11}$$

Therefore, by using the fact $g'' > 0$ and at least one $\varepsilon_i \neq 0$ we conclude that $|\det(DR)| > 1$. Since $R(A) \subset A$, $|\det(DR)| > 1$ implies A has measure zero.

We have shown that the set of initial conditions which live forever without any absorptions has a Lebesgue measure zero. To complete our proof on the problem of synchronization with threshold effects, we turn now to the discussion of the set of initial conditions which allow absorptions. What we need to show is that the set of initial conditions which, after a finite number of absorptions, live forever without ending in synchrony has a measure zero.

Due to the process of absorptions, we have to introduce a new notation. The state of the N oscillator was previously denoted by $\phi = (\phi_1, \ldots, \phi_n)$ with $n = N-1$ and $\phi \in S$ with S defined in (3.5). We now replace S with S_n, the subscript n showing explicitly the number of oscillators. When absorption occurs, the number of variables for describing the system gets reduced. Suppose that $\phi \in S_n$ gets absorbed by h_1 to S_k; what it means is that the oscillators corresponding to $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{n-k+1}$ have been absorbed by the oscillator corresponding ε_1. These oscillators thus form a group which moves together with a combined pulsed strength $\varepsilon_1+\varepsilon_2+\cdots\varepsilon_{n-k+1}$. The new state vector $\phi \in S_k$ is now denoted by $\phi=(\phi_1,\phi_2,\ldots,\phi_k)$ with a new pulse strength

$$\varepsilon_1'=\varepsilon_{n-k+2},\ldots,\varepsilon_{k-1}'=\varepsilon_n,\varepsilon_k'=\varepsilon_1+\ldots+\varepsilon_{n-k+1} .$$

This absorption process continues until it reaches synchrony ($k=0$) or gets stuck forever at some stage with $k>0$. Let us define B as the set of initial conditions in S_n which, upon iterations of h_i with the ε sequence $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n+1}$, never reach synchrony. Then we prove the following theorem on B.

Theorem 5. The set B has Lebesgue measure zero.

Proof. We will prove this theorem by induction. Suppose the theorem is true for all ε sequences $\varepsilon_1,\ldots,\varepsilon_k$ on S_k, with $k<n$. Let $B_{r,k}$ be the set of $\phi \in B$ such that ϕ survives the applications of $h_1,\ldots,h_{r-1}$, and then gets absorbed by h_r to S_k. Thus

$$B=A \cup \bigcup_{r\geq 1;1\leq k\leq n} B_{r,k} . \tag{3.12}$$

Since the measure of A is zero, in order to show that the measure of B vanishes, we have to establish the fact that the measure of each $B_{r,k}$ is zero. Our proof is identical to the one given by Mirollo and Strogatz [2]. Let us start with $B_{1,k}$ which consists of points absorbed by h_1 to S_k. By induction, these points must be absorbed into a set Z in S_k where Z has measure zero. This is due to the fact that, in any problem in S_k, the set of points which do not reach synchrony has measure zero. Suppose $\phi=(\phi_1,\ldots,\phi_n)\in B_{1,k}$ and Z are the set of measure zero. Under the application of h_1 we have

$$h_1(\phi)=[\sigma_1,\sigma_2,\ldots,g(\varepsilon_1+f(\sigma_l)),g(\varepsilon_1+f(\sigma_{l+1})),\ldots,g(\varepsilon_1+f(\sigma_k))]\in Z , \tag{3.13}$$

where we have explicitly assumed some of the σ_i are below ϕ_c (for our case $\sigma_1,\ldots,\sigma_l<\phi_c$). $\sigma(\phi)$ is defined as $\sigma(\phi)=(\sigma_1,\ldots,\sigma_n)$. Hence $(\sigma_1,\ldots,\sigma_k)\in\tau_1^{-1}Z$, where τ_1 is the map acting on S_k. It is obvious that τ_1 is a diffeomorphism, therefore the measure of $\tau_1^{-1}Z$ vanishes. This implies that the projection of $\sigma B_{1,k}$ to S_k has zero measure. Since σ is also a diffeomorphism, we conclude that the measure of $B_{1,k}$ is zero.

For $r>1$, we have

$$h_{r-1}h_{r-2}\cdots h_1 B_{r,k}\subset B_{1,k} . \tag{3.14}$$

Since each h_i is a diffeomorphism, hence $B_{r,k}$ has zero measure for all k, $r>1$.

IV. CONCLUSIONS

For the integrate and fire models, we have shown that, by including threshold effects, synchronization can also occur for almost all initial conditions when the threshold condition ϕ_c is less than $\frac{1}{2}$. For the two-oscillator system, as $\phi_c>\frac{1}{2}$, we explicitly show that there exists a window $(1-\phi_c,\phi_c)$ in which synchronization does not happen. In fact, for any $\phi\in(1-\phi_c,\phi_c)$, the return map R is an identity mapping, $R(\phi)=\phi$. Intuitively, it is easy to see that when ϕ_c get close to 1, the possibility of having synchronization is unfavorable, since $\phi_c=1$ always implies $R(\phi)=\phi$. Furthermore, when $\phi_c=0$, it is shown by Mirollo and Strogatz that almost all initial conditions end up in synchrony. Thus, it is no surprise that the same result can also occur as ϕ_c moves away from zero. This work provides the analytical proof for arbitrary ϕ_c. As pointed out in [2], this kind of model neglects the spatial structure in the discussion. Also, fluctuation effects such as temperature, etc., are not included. Therefore the question of stability should be addressed. We hope to return to these questions in the future.

ACKNOWLEDGMENTS

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